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Energetics of two circular inclusions in anti-plane elastostatics

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Abstract

In this paper, we use the recently derived solution of two circular elastic inclusions under anti-plane shear deformation (Honein et al., 1992a, b) to evaluate the material forces, as well as expanding and rotating moments 'acting' on inclusions. These may be defined as the energy changes (e.g. energy release rates) accompanying unit translation, self-similar expansion and rotation of inclusions, respectively. The bond between the inclusions and the matrix is assumed to be perfect and the calculation is performed using the concept of the J, M and L path-independent integrals, respectively. The results obtained are valid under arbitrary loading.

An illustrative example shows that two circular holes under remote uniform shear stress attract each other and that the J and M integrals grow without bound as the two holes become infinitely close. A careful examination of the expression for these integrals yields the result that the J and M integrals tend to infinity proportionally to $1/\sqrt{\varepsilon}$, where ε is a non-dimensional distance between the holes. It is also noticed that the J integral decays rapidly to zero as the two holes become four or five radii apart. Other examples of two circular holes and inclusions under various stress fields are also considered and discussed. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

In the study of defect mechanics in solids, conservation laws, in the form of path-independent integrals, play a fundamental role. These integrals can be related to total potential energy changes (e.g. energy release rates) as a defect translates, rotates or expands self-similarly in solids.[1] For example, the

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[1] To the authors' knowledge, this statement is proved in the open literature for traction-free cavity only but its validity extends to 'floating' inclusions.

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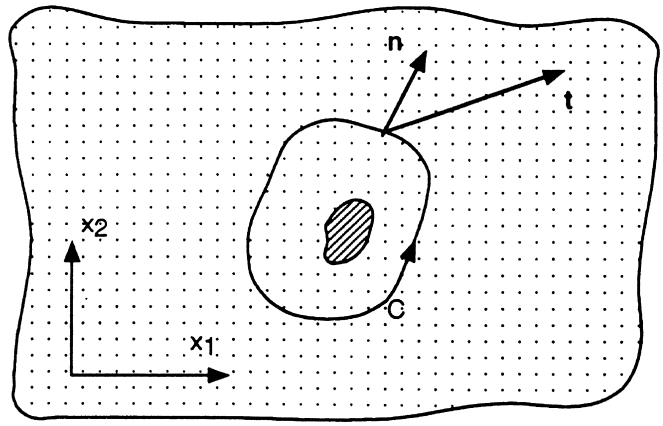

Fig. 1. A contour surrounding a defect illustrating path-independent integrals.

classical J integral (Rice, 1968), given by

$$J = \oint_C W \, \mathrm{d}x_2 - t_i u_{i,1} \, \mathrm{d}l, \tag{1}$$

where, in two-dimensional space, C is a closed curve in the x_1, x_2 plane, W is the strain energy density, t_i ($i = 1, 2$) are the components of traction acting on the boundary of the region enclosed by C, u_i are the components of the displacement field, and $\mathrm{d}l$ is the infinitesimal arc length along the curve C (see Fig. 1), can be interpreted as the negative of the potential energy release rate as a traction-free cavity undergoes a unit translation in the x_1-direction (see Budiansky and Rice, 1973).

The J integral is actually the first component of the vector

$$J_k = \oint_C \left(W n_k - t_i u_{i,k} \right) \mathrm{d}l = \oint_C b_k \, \mathrm{d}l, \tag{2}$$

where n_k is the unit outward normal to C, lying in the same plane and $\mathrm{d}l$ can be written as $\mathrm{d}l = -n_2 \, \mathrm{d}x_1 + n_1 \, \mathrm{d}x_2$. The J_2 integral is interpreted as the negative of the total potential energy release rate as the traction-free cavity undergoes a unit translation in the x_2-direction.

Other path-independent integrals derived by Günther (1962) and Knowles and Sternberg (1972) include the L and M integrals given in two dimensions by

$$L = \oint_C \varepsilon_{3ij} \left(W x_j n_i + t_i u_j - t_k u_{k,i} x_j \right) \mathrm{d}l = \oint_C \varepsilon_{3ij} \left(b_i x_j + t_i u_j \right) \mathrm{d}l, \tag{3}$$

and

$$M = \oint_C \left(W x_i n_i - t_k u_{k,i} x_i \right) \mathrm{d}l = \oint_C b_k x_k \, \mathrm{d}l, \tag{4}$$

where ε_{ijk} is the alternating tensor. The L integral may be called the material rotating moment and can be interpreted as the positive of the energy change as a traction-free cavity undergoes a unit rotation with respect to the origin, while the M integral may be called the expanding moment (virial) and can be interpreted as the negative of the energy change as a traction-free cavity undergoes a self-similar expansion relative to the origin.

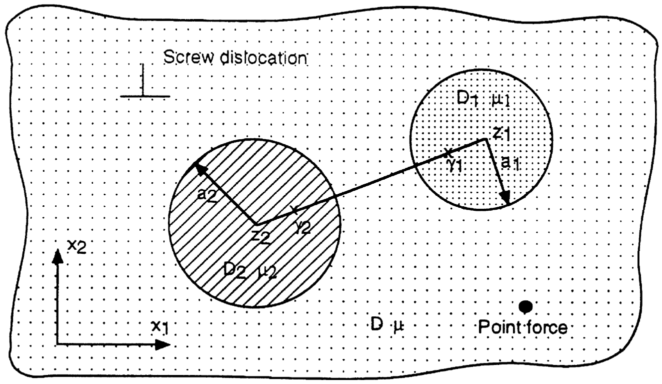

Fig. 2. Two circular inclusions under arbitrary anti-plane deformation.

In this paper, these path-independent integrals are calculated around a circular elastic inclusion interacting with another one under anti-plane shear deformation (see Fig. 2) and interpreted in terms of energy changes as the inclusion enclosed by the path of integration undergoes either a unit translation, rotation or self-similar expansion. The inclusions considered are assumed to be perfectly bonded to a matrix, of infinite extent, subjected to arbitrary loading. The formulae obtained are universal, i.e. independent of the loading or singularities, and the asymptotic behavior as two holes approach each other is investigated.

2. Two circular inclusions in anti-plane elastostatics

Under anti-plane deformation, the displacement field satisfies

$$u_1 = u_2 \equiv 0, \quad u = u_3 = u_3(x_1, x_2), \tag{5}$$

i.e., the only nonvanishing component of displacement, with respect to a Cartesian coordinate system $Ox_1x_2x_3$, is $u = u_3$ which is a function of the coordinates x_1 and x_2 only.

As is well known, the displacement field u can be given, in the case of anti-plane elastostatics, in terms of an analytic complex function ϕ of a complex variable $z = x_1 + ix_2$, namely

$$u = \frac{1}{\mu}\Im\{\phi(z)\}, \tag{6}$$

where μ is the shear modulus and $\Im$ stands for the imaginary part of the argument. Then the stress field, in Cartesian coordinates, is related to ϕ by

$$\sigma_{23} + i\sigma_{13} = \phi', \tag{7}$$

and, in polar coordinates, by

$$\sigma_{3\theta} + i\sigma_{3r} = e^{i\theta}\phi', \tag{8}$$

where, throughout this paper, a prime indicates differentiation with respect to the complex variable z. Thus, any analytical function ϕ will lead to a stress field while satisfying compatibility and equilibrium.

In the following, we recall briefly some results derived previously by the authors (Honein et al., 1992a, b). Consider the system shown in Fig. 2, consisting of three domains D, D_1 and D_2, where D is of infinite extent and is occupied by a material (the matrix), of shear modulus μ, which is subjected to arbitrary loading (singularities).

The regions D_1 and D_2 are circular domains centered at z_1 and z_2, of radii a_1 and a_2 and occupied by materials of shear moduli μ_1 and μ_2, respectively. No loading is applied inside D_1 and D_2.

We assume that the arbitrary singularities would produce the complex potential ϕ if D occupied the whole space. We seek the solution of the heterogeneous problem in the form:

$$\text{in } D: \quad \Phi = \phi + \mathscr{H}_1(f_1) + \mathscr{H}_2(f_2), \tag{9}$$

$$\text{in } D_1: \quad \Phi_1 = \phi + \phi_1, \tag{10}$$

$$\text{in } D_2: \quad \Phi_2 = \phi + \phi_2, \tag{11}$$

where ϕ_1 and f_1 are analytic in D_1, while ϕ_2 and f_2 are analytic in D_2.

Here $\mathscr{H}_i$ (i = 1, 2) designates the 'hat' transformation with respect to the circle $C_i = \partial D_i$ bounding D_i, i.e., $\mathscr{H}_i$ is defined by

$$\mathscr{H}_i(f_i)(z) = \overline{f_i(A_i z)}, \tag{12}$$

where the overbar indicates complex conjugation and A_i is the inversion with respect to C_i. We recall that A_i is given by

$$A_i z = \frac{a_i^2}{\bar{z} - \bar{z}_i} + z_i, \tag{13}$$

The continuity of tractions across ∂D_1 and ∂D_2 leads to

$$f_1 - \alpha_1 \mathscr{H}_2(f_2) = \alpha_1 \phi + \text{constant}, \tag{14}$$

and

$$f_2 - \alpha_2 \mathscr{H}_1(f_1) = \alpha_2 \phi + \text{constant}, \tag{15}$$

where $\alpha_i = (\mu_i - \mu)(\mu_i + \mu)^{-1}$ and constant refers to complex numbers.

The governing equations for f_1 and f_2 are obtained as:

$$f_1(z) - \alpha_1 \alpha_2 f_1(Mz) = \alpha_1 \phi + \alpha_1 \alpha_2 \overline{\phi(A_1 z)} + \text{constant}, \tag{16}$$

and

$$f_2(z) - \alpha_1 \alpha_2 f_2(Nz) = \alpha_2 \phi + \alpha_1 \alpha_2 \overline{\phi(A_2 z)} + \text{constant}, \tag{17}$$

where $Mz = A_1 A_2 z$ and $Nz = A_2 A_1 z = M^{-1} z$.

By effecting the change of variables

$$w_1 = T_1 z = \frac{z - \gamma_1}{z - \gamma_2}, \quad w_2 = T_2 z = \frac{z - \gamma_2}{z - \gamma_1}, \tag{18}$$

where γ_1 and γ_2 are the fixed points of the transformation M (Honein et al., 1992b), and

$$g_1 = f_1 \circ T_1^{-1}, \quad g_2 = f_2 \circ T_2^{-1}, \tag{19}$$

we obtain

$$g_1(w_1) - \alpha_1\alpha_2 g_1(k_1k_2w_1) = \alpha_1\phi + \alpha_1\alpha_2\overline{\phi(A_2z)} + \text{constant} = R_1(w_1), \tag{20}$$

and

$$g_2(w_2) - \alpha_1\alpha_2 g_2(k_1k_2w_2) = \alpha_2\phi + \alpha_1\alpha_2\overline{\phi(A_1z)} + \text{constant} = R_2(w_2), \tag{21}$$

where $k_i = T_i(z_i)$, $i = 1, 2$ and the operation $\circ$ composites is defined so that $f \circ T(z) = f(T(z))$.

3. Evaluation of the J integral

In terms of the complex potential Φ defined by eqn (6), the J_1 and J_2 integrals have been shown by Budiansky and Rice (1973) to be given by

$$J_1 - iJ_2 = -\frac{i}{2\mu}\oint_C (\Phi')^2\,\mathrm{d}z, \tag{22}$$

where C is a closed contour surrounding the defect. Our purpose is to find the material force[2] acting on a circular inclusion interacting with another one under arbitrary external loading (singularities). We take, without loss of generality, the x_1-axis along the line joining the centers of the two inclusions. We would like to express the result in terms of the elastic field which would exist in the homogeneous body when the two elastic inclusions are absent. To this end, we use the solution derived in Honein et al. (1992b).

We wish to evaluate the material force acting on the first inclusion. Thus, we replace C by ∂D_1 and evaluate, using eqn (9), the integral

$$\oint_{\partial D_1} (\Phi')^2\,\mathrm{d}z = \oint_{\partial D_1} \Big[\big(\phi' + \mathscr{H}_2(f_2)'\big)^2 + \big(\mathscr{H}_1(f_1)'\big)^2 + 2\big(\phi' + \mathscr{H}_2(f_2)'\big)\mathscr{H}_1(f_1)'\Big]\,\mathrm{d}z. \tag{23}$$

But $\phi' + \mathscr{H}_2(f_2)'$ is analytic inside D_1, therefore, its contribution to the integral is zero by Cauchy's theorem. Similarly we can show that the contribution of $(\mathscr{H}_1(f_1)')^2$ is zero by applying the residue theorem. Thus the integral reduces to

$$\oint_{\partial D_1} (\Phi')^2\,\mathrm{d}z = 2\oint_{\partial D_1} \big(\phi' + \mathscr{H}_2(f_2)'\big)\mathscr{H}_1(f_1)'\,\mathrm{d}z$$

$$= \frac{2}{\alpha_1}\oint_{\partial D_1} f_1'\mathscr{H}_1(f_1)'\,\mathrm{d}z, \tag{24}$$

where in the last substitution we have used eqn (14).

With the change of variable given by the first of Eqs. (18), the integral can be written over the transformed contour $T_1(\partial D_1)$ as

[2] Here, by material force is meant the change in the total potential energy of the system as the inclusion undergoes a unit translation.

$$\oint_{\partial D_1} (\Phi')^2 \, \mathrm{d}z = \frac{2}{\alpha_1(\gamma_1 - \gamma_2)} \oint_{T_1(\partial D_1)} \frac{\mathrm{d}}{\mathrm{d}w_1} g_1(w_1) \frac{\mathrm{d}}{\mathrm{d}w_1} \overline{g_1(k_1/\bar{w}_1)} (w_1 - 1)^2 \, \mathrm{d}w_1. \tag{25}$$

The function g_1, which is analytic for $|w_1| < k_1^{1/2}$, is the solution to the governing equation

$$g_1(w_1) - \alpha_1\alpha_2 g_1(k_1 k_2 w_1) = \alpha_1 \phi \circ T_1^{-1}(w_1) + \alpha_1\alpha_2 \overline{\phi \circ T_1^{-1}\left(\frac{1}{k_2 \bar{w}_1}\right)}, \tag{26}$$

which is deduced from (16) after dropping the constant.

The function $\phi \circ T_1^{-1}$ is known and has the expansion

$$\phi \circ T_1^{-1}(w_1) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} A_n w_1^n, & \text{for } |w_1| < k_1^{1/2}; \\ \displaystyle\sum_{n=0}^{\infty} B_n w_1^{-n}, & \text{for } |w_1| > k_1^{-1/2} \end{cases} \tag{27}$$

The solution of (26) can then be written as

$$g_1(w_1) = \sum_{n=0}^{\infty} \alpha_1 u_n w_1^n, \tag{28}$$

with u_n given by

$$u_n = \frac{A_n + \alpha_2 \bar{B}_n k_2^n}{1 - \alpha_1\alpha_2 (k_1 k_2)^n}. \tag{29}$$

On substituting eqn (28) into (25) we obtain

$$\oint_{\partial D_1} (\Phi')^2 \, \mathrm{d}z = \frac{2\alpha_1}{(\gamma_2 - \gamma_1)} \oint_{T_1(\partial D_1)} \sum_{n=1}^{\infty} n u_n w_1^{n-1} \sum_{m=1}^{\infty} m \bar{u}_m k_1^m w_1^{-m-1} (w_1 - 1)^2 \, \mathrm{d}w_1, \tag{30}$$

which yields, after using the residue theorem

$$\oint_{\partial D_1} (\Phi')^2 \, \mathrm{d}z = \frac{4\pi\alpha_1 i}{\gamma_2 - \gamma_1} \sum_{n=1}^{\infty} n u_n k_1^{n-1} \left[(n+1)\bar{u}_{n+1} k_1^2 - 2n\bar{u}_n k_1 + (n-1)\bar{u}_{n-1} \right]. \tag{31}$$

Finally the result for the J_1 and J_2 integrals is obtained as

$$J_1 - iJ_2 = \frac{2\pi\alpha_1}{\mu(\gamma_2 - \gamma_1)} \sum_{n=1}^{\infty} n u_n k_1^{n-1} \left[(n+1)\bar{u}_{n+1} k_1^2 - 2n\bar{u}_n k_1 + (n-1)\bar{u}_{n-1} \right] \tag{32}$$

where we recall that u_n is given by eqn (29).

We note that for $\alpha_1 = 0$ (no inclusion inside D_1) the J_1 and J_2 integrals vanish as they must in this case. The series in eqn (32) is a geometric series with the leading term of the order of nk_1^n and is rapidly convergent since $|k_1| < 1$.

eqn (32) is a universal expression giving the material force acting on a circular inclusion interacting with another one in terms of the elastic field that would exist if the two inclusions were absent and the entire plane was occupied by the matrix material (the corresponding homogeneous problem).

In the following, some special cases of loading will be worked out for illustrative purposes.

Furthermore, the asymptotic behavior of the material force as the two holes approach each other under remote uniform shear stress will be indicated.

3.1. Examples

3.1.1. Two circular inclusions under uniform shear

In this case the complex potential for the homogeneous problem is given by $\phi = \tau z$ where τ is a real constant, and the stress field is given by $\sigma_{23} = \tau = \phi'$. We readily obtain the expansion

$$\phi \circ T_1^{-1}(w_1) = \begin{cases} \tau(\gamma_1 - \gamma_2 w_1)\sum_{n=0}^{\infty} w_1^n, & \text{for } |w_1| < 1; \\ \tau\left(\gamma_2 - \gamma_1 w_1^{-n}\right)\sum_{n=0}^{\infty} w_1^{-n}, & \text{for } |w_1| > 1. \end{cases} \tag{33}$$

From which we deduce by inspection that

$$A_n = \tau(\gamma_1 - \gamma_2) = -B_n \quad \text{for } n \geqslant 1; \tag{34}$$

and thus,

$$u_n = \frac{\tau(\gamma_1 - \gamma_2)\left(1 - \alpha_2 k_2^n\right)}{1 - \alpha_1\alpha_2 k_1^n k_2^n}. \tag{35}$$

The formula (32) for the J_1 and J_2 integrals becomes

$$\begin{aligned} j_1 - iJ_2 = \frac{2\pi\alpha_1(\gamma_2 - \gamma_1)\tau^2}{\mu}\sum_{n=1}^{\infty}\frac{n\left(1 - \alpha_2 k_2^n\right)}{1 - \alpha_1\alpha_2 k_1^n k_2^n}k_1^{n-1}\Bigg[\frac{(n+1)\left(1 - \alpha_2 k_2^{n+1}\right)}{1 - \alpha_1\alpha_2 k_1^{n+1}k_2^{n+1}}k_1^2 \\ - 2n\frac{\left(1 - \alpha_2 k_2^n\right)}{1 - \alpha_1\alpha_2 k_1^n k_2^n}k_1 + (n-1)\frac{\left(1 - \alpha_2 k_2^{n-1}\right)}{1 - \alpha_1\alpha_2 k_1^{n-1}k_2^{n-1}}\Bigg]. \end{aligned} \tag{36}$$

It can be verified that, when $\alpha_2 = 0$, i.e. when the second inclusion is absent, eqn (36) yields the expected result $J_1 - iJ_2 = 0$.

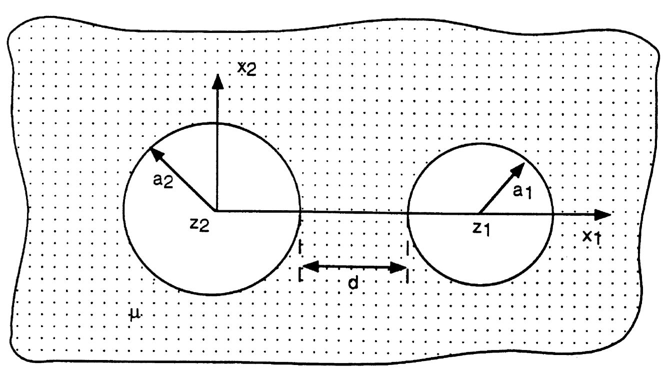

Fig. 3. Two circular holes centered on the x_1-axis. The second hole is centered at the origin.

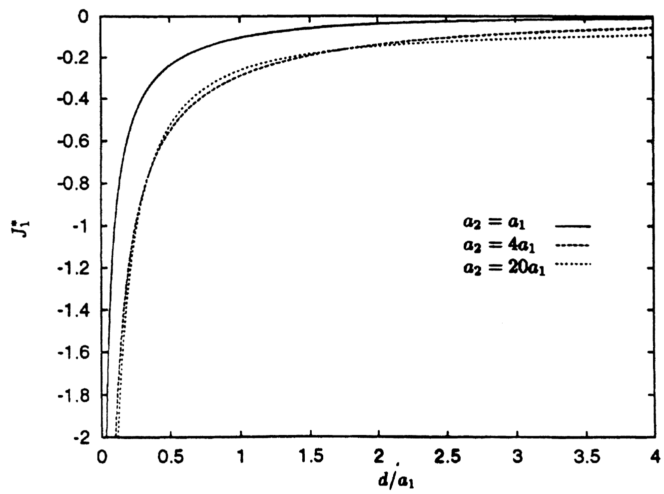

Fig. 4. Graph of J_1^*, the non-dimensional J_1 integral, as a function of the non-dimensional distance of separation $d^* = d/a_1$ of the two holes of Fig. 3 under remote uniform shear.

The right-hand side of eqn (36) is real and therefore the J_2-integral is zero, as would be expected on physical grounds.

In Fig. 3, we present two circular cavities of radii a_1 and a_2 and centered at the x_1-axis at z_1 and z_2 (the origin), respectively. The two holes are separated by a distance d. The graph for the non-dimensional J_1-integral, $J_1^* = \mu J_1/(2\pi a_1 \tau^2)$, is sketched in Fig. 4 as a function of the non-dimensional distance of separation $d^* = d/a_1$ of the two holes of Fig. 3.

It is seen that J_1 grows without bound as d^* tends to zero.

By performing an asymptotic study of the series as $d^* \to 0$ we can readily show that $|J_1| \sim (\pi\tau^2/\mu\sqrt{\varepsilon})a_{\mathrm{H}}$ as $d^* \to 0$ where $\varepsilon = d/a_{\mathrm{H}} = a_1 d^*/a_{\mathrm{H}}$, a_{H} being the harmonic average of the radii a_1 and a_2, given by $a_{\mathrm{H}} = 2a_1a_2/(a_1 + a_2)$.

In Fig. 5, the case of two circular inclusions having $\mu_1 = 0.1\mu$ and $\mu_2 = 10\mu$ under uniform shear is considered. For this case, we sketch in Fig. 6 the non-dimensional J-integral as a function of d^*, the non-dimensional distance between the two inclusions, for different values of a_2/a_1. In this case, the material force between the inclusions is repulsive and it tends to zero as the inclusions become far apart. It is worthwhile noting that because of the shielding effect, the repulsive material force is not maximum

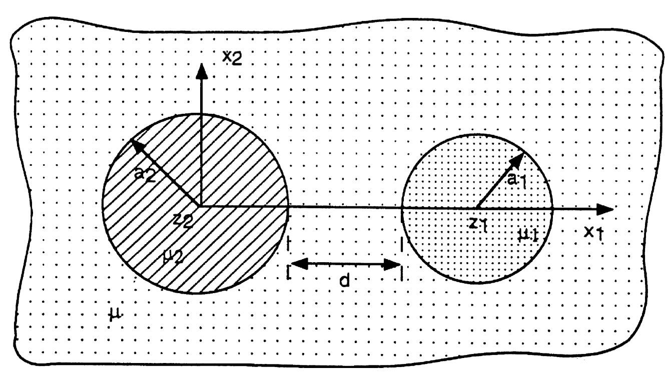

Fig. 5. Two circular inclusions centered on the x_1-axis. The second inclusion is centered at the origin.

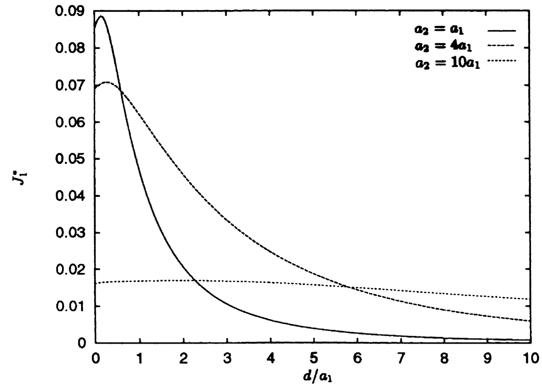

Fig. 6. Graph of J_1^*, the non-dimensional J_1 integral, as a function of the distance of separation $d^* = d/a_1$ of the two inclusions of Fig. 5 under uniform shear.

when the inclusions touch each other. This maximum occurs when the inclusions are close to each other ($d/a_1 < 0.5a_2/a_1$).

3.1.2. Two circular inclusions under linearly distributed shear

In this case, the complex potential for the homogeneous problem is given by $\phi = \tau z^2$ where τ is a complex constant and the stress field is given by $\sigma_{23} + i\sigma_{13} = 2\tau z$. eqn (33) should be replaced by

$$\phi \circ T_1^{-1}(w_1) = \begin{cases} \tau(\gamma_1 - \gamma_2 w_1)^2 \sum_{n=0}^{\infty} (n+1) w_1^n, & \text{for } |w_1| < 1; \\ \tau\left(\gamma_2 - \gamma_1 w_1^{-1}\right)^2 \sum_{n=1}^{\infty} (n-1) w_1^{-n}, & \text{for } |w_1| < 1. \end{cases} \tag{37}$$

From which we deduce that

$$A_1 = 2\tau\gamma_1(\gamma_1 - \gamma_2) \tag{38}$$

$$A_n = \tau\left[(n+1)\gamma_1^2 - 2n\gamma_1\gamma_2 + (n-1)\gamma_2^2\right], \quad \text{for } n \geqslant 2. \tag{39}$$

The coefficients B_i are obtained from A_i by interchanging the subscripts 1 and 2.

We can also verify that the following relations hold:

$$A_{n+1} - A_n = B_{n+1} - B_n = \tau(\gamma_1 - \gamma_2)^2. \tag{40}$$

In Fig. 7, we sketch the graph of J_1^* as a function of the non-dimensional separation d^* of two holes as depicted in Fig. 3 with $a_2 = 2a_1$. It is seen that the J_1-integral vanishes at a certain distance denoted by d_c and that Hole 1 of radius a_1 is attracted to the second hole which is centered at the origin if $d^* < d_c$, and it is repulsed if $d^* > d_c$. Furthermore, for large values of d^*, J_1^* increases linearly with the non-dimensional distance of separation d^* asymptotically to the solution of a single hole as given by eqn (41) below.

Under the present loading, the J_2 component acting on Hole 1 vanishes along the x_1-axis. Therefore, for $d = d_c$ the material force acting on Hole 1 has vanishing components in both directions. Hence, this position is an equilibrium position, however, this equilibrium is unstable

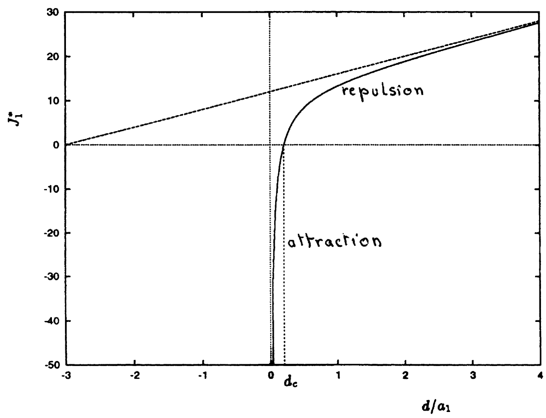

Fig. 7. Graph of J_1^*, the non-dimensional J_1 integral, as a function of the distance of separation $d^* = d/a_1$ of the two holes of Fig. 3, with $a_2 = 2a_1$, and under remote linearly distributed shear. For $d^* > d_c$, the material force between the holes is repulsive, and it is attractive for $d^* < d_c$. The straight line represents the material force on a single hole as given by eqn (41).

because any small deviation from this position will generate a material force which will pull the hole further away form it.

In order to represent graphically how the material force changes as the location of Hole 1 varies, we fix the center of Hole 2 at the origin with $a_2 = 2a_1$. At each point z, we can compute the material force when Hole 1 is centered at z. This will be represented by a bound vector with tail at z. These non-dimensional arrows are sketched in Fig. 8 and give a visualization of the influence of Hole 2 in its neighborhood on Hole 1. It can be seen from the figure that the two holes attract each other when they are sufficiently close or when Hole 1 is close to the x_2-axis.

In order to quantify the effect of the presence of Hole 2 on the material force, we compute analytically the components of the J-integral acting on Hole 1 when Hole 2 is absent. In this case, since the elastic field is not uniform, the material force does not vanish and is given by

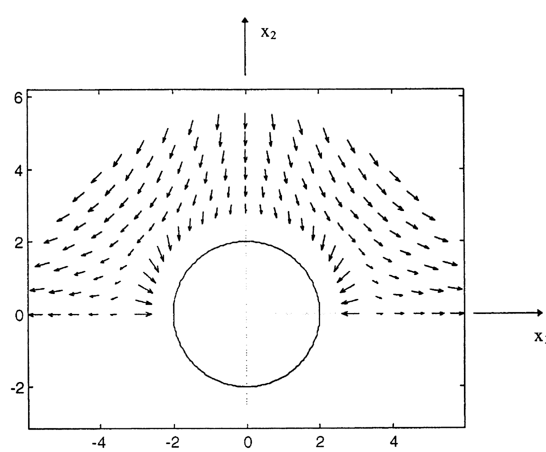

Fig. 8. The visual representation of the material force between two holes. Hole 2 is shown and is centered at the origin. Each bound vector (arrow) represents the non-dimensional material force acting on Hole 1 (not shown) when it is centered at the tail of the arrow. The two holes are under remote linearly distributed shear field.

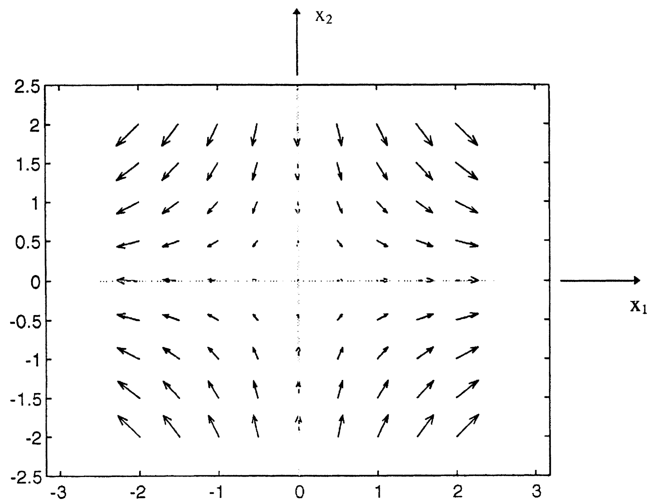

Fig. 9. The absence of Hole 2 influences the material forces acting on Hole 1. A comparison with Fig. 8 shows that the influence is local and decays rapidly.

$$J_1 - iJ_2 = \frac{2\pi}{\mu}\left(- a_1 \alpha a_1^2 \bar{z}_1 |\tau|^2\right) \tag{41}$$

where z_1 is the center of the inclusion and a_1 is its radius. These are represented in Fig. 9 by bound vectors emanating at the center z_1 of Hole 1. By comparing Figs. 8 and 9, it can be seen that the effect of Hole 2 when placed at the origin is strong, but is local and decays rapidly as one moves away from the origin.

4. Evaluation of L and M integrals

The evaluation of the L and M integrals proceeds along similar lines. We note in this regard that the formula given by Budiansky and Rice (1973) for L and M integrals contains a misprint and must read

$$L - iM = -\frac{1}{2\mu}\oint_{\partial D_1} (z - z_1)(\Phi')^2 \, \mathrm{d}z, \tag{42}$$

with the minus sign multiplying the imaginary unit instead of the plus sign given in their formula. The center of ∂D_1, z_1, is the point with respect to which L and M are evaluated. Then for traction-free holes L can be interpreted as the positive of the total potential energy released as the defect undergoes a unit rotation with respect to the center z_1, while M is the negative of the total potential energy released as the defect undergoes a self-similar expansion. The L integral given by (42) vanishes, due to symmetry, as expected. We show shortly, as a check of our result, that this is indeed the case.

On substituting (9) in (42), we can show by applying the residue theorem that the contribution of $(z - z_1)\,(\phi' + \mathscr{H}_2(f_2)')^2$ and $(z - z_1)\,(\mathscr{H}_1(f_1)')^2$ to the integral vanishes and we are left with

$$\oint_{\partial D_1} (z - z_1)(\Phi')^2 \, \mathrm{d}z = \frac{2}{\alpha_1}\oint_{\partial D_1} (z - z_1) f_1' \mathscr{H}_1(f_1)' \, \mathrm{d}z. \tag{43}$$

In terms of the variable w_1 defined by (18), the integral becomes

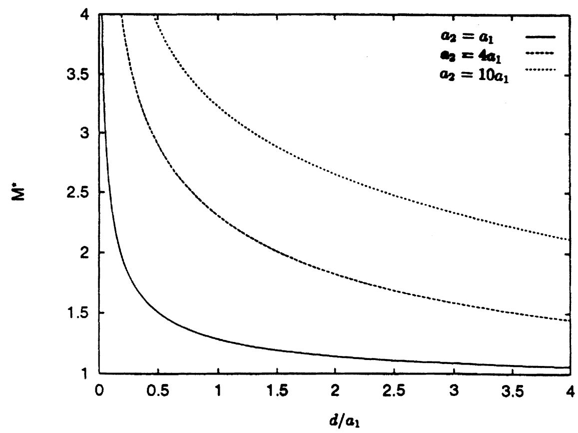

Fig. 10. Graph of M^*, the non-dimensional M integral, as a function of the distance of separation $d^* = d/a_1$ of the two holes of Fig. 3 under remote uniform shear.

$$\oint_{\partial D_1} (z - z_1)(\Phi')^2 \, dz = \frac{2}{\alpha_1(\gamma_1 - \gamma_2)} \oint_{T_1(\partial D_1)} (w_1 - k_1)(w_1 - 1) \frac{d}{dw_1} g_1(w_1) \frac{d}{dw_1} \overline{g_1(k_1/\bar{w}_1)} \, dw_1. \tag{44}$$

On replacing g_1 by its value given by eqn (28) and using the residue theorem we obtain

$$\oint_{\partial D_1} (z - z_1)(\Phi')^2 \, dz = \frac{2\pi\alpha_1 i}{1 - k_1} \sum_{n=1}^{\infty} n k_1^n \left[2(n+1) k_1 \Re(u_n \bar{u}_{n+1}) - n(1 + k_1)|u_n|^2 \right]. \tag{45}$$

Hence we obtain by (42) $L = 0$ as expected, and

$$M = \frac{2\pi\alpha_1}{\mu(1 - k_1)} \sum_{n=1}^{\infty} n k_1^n \left[n(1 + k_1)|u_n|^2 - 2(n+1) k_1 \Re(u_n \bar{u}_{n+1}) \right]. \tag{46}$$

This last expression represents a universal formula giving the M integral in terms of the elastic field of the corresponding homogeneous problem.

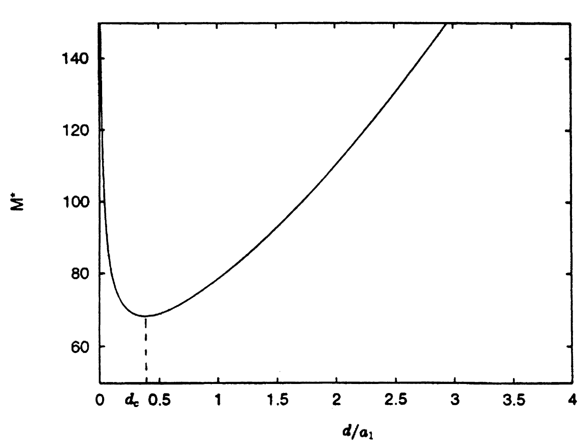

Fig. 11. Graph of M^*, the non-dimensional M integral, as a function of the distance of separation $d^* = d/a_1$ of the two holes of Fig. 3, with $a_2 = 2a_1$, and under remote linearly distributed shear.

The non-dimensional M integral, $M^* = (\mu M/2\pi\tau^2)$, is plotted in Fig. 10 as a function of the non-dimensional distance of separation d^* of two holes of different radii, subjected to uniform remote longitudinal shear τ. We see that M, like J_1, grows without bound as d^* tends to zero. We can show that the asymptotic behavior of M, as $d^* \to 0$, is given by

$$M \sim \frac{\pi\tau^2}{\mu\sqrt{\varepsilon}} a_{\mathrm{H}} a_1 \sim a_1 J_1. \tag{47}$$

For the case of linearly distributed loading and the same configuration as in example 3.1.2, M^* is drawn in Fig. 11 as a function of d^*. We see that as the two cavities approach each other, the expanding moment M^* becomes very large due to the interaction of the cavities. Also, as Hole 1 moves on the x_1-axis away from the origin, M^* increases indefinitely due to the external linearly distributed loading. At a distance $d^* = d_c$, M^* possesses a minimum, i.e. the expanding moment acting on the cavity is minimum at this location.

5. Conclusions

In this paper, we considered an extended elastic medium containing two perfectly bonded circular inclusions and subjected to arbitrary singularities (physical forces and/or moments) inducing an anti-plane strain field. We evaluated the material forces and moments (i.e. energy change rates due to translation, rotation and self-similar expansion) 'acting' on one inclusion by deriving explicit expressions for the J_1, J_2, L and M integrals. Two specific cases of uniform and linearly varying remote shear stress fields were considered as particular illustrative examples. In the latter case, when the inclusions become cavities, the material force may be attractive or repulsive depending on the distance of separation. In the former case, the material force tends to infinity as $1/\sqrt{\varepsilon}$ when the two cavities approach each other, where ε is the non-dimensional distance separating the cavities.

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